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Renormalization of a Hard-Core Guest Charge Immersed in a Two-Dimensional Electrolyte

L. Šamaj[1]

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This paper is a continuation of a previous one [L. Šamaj, *J. Stat. Phys.* **120**:125 (2005)] dealing with the renormalization of a guest charge immersed in a two-dimensional logarithmic Coulomb gas of pointlike ± unit charges, the latter system being in the stability-against-collapse regime of reduced inverse temperatures $0 \leq \beta < 2$. In the previous work, using a sine-Gordon representation of the Coulomb gas, an exact renormalized-charge formula was derived for the special case of the *pointlike* guest charge Q, in its stability regime $\beta|Q| < 2$. In the present paper, we extend the renormalized-charge treatment to the guest charge with a hard core of radius σ, which allows us to go beyond the stability border $\beta|Q| = 2$. In the limit of the hard-core radius much smaller than the correlation length of the Coulomb-gas species and at a strictly finite temperature, due to the counterion condensation in the extended region $\beta|Q| > 2$, the renormalized charge Q_{ren} turns out to be a periodic function of the bare charge Q with period 1. The renormalized charge therefore does not saturate at a specific finite value as $|Q| \to \infty$, but oscillates between two extreme values. In the high-temperature Poisson-Boltzmann scaling regime of limits $\beta \to 0$ and $Q \to \infty$ with the product βQ being finite, one reproduces the Manning-Oosawa type of counterion condensation with the uniform saturation of βQ_{ren} at the value $4/\pi$ in the region $\beta|Q| \geq 2$. The obtained results disprove the "regularization hypothesis" of the previous work about the possibility of an analytic continuation of the formula for Q_{ren} from the stability region $\beta|Q| < 2$ to $\beta|Q| \geq 2$.

KEY WORDS: Coulomb systems; logarithmic interactions; Sine-Gordon model; renormalized charge; counterion condensation.

1. INTRODUCTION

The concept of renormalized charge is of primary importance in the equilibrium statistical mechanics of colloids, see, e.g., refs. (1–5). This concept is based

[1] Institute of Physics, Slovak Academy of Sciences, Dúbravská cesta 9, 845 11 Bratislava, Slovak Republic; e-mail: fyzimaes@savba.sk

on the assumption that, at a finite temperature, the electric potential induced by a "guest" (say colloidal) charged particle, immersed in an infinite electrolyte, exhibits, at large distances from this particle, basically the screening form given by the high-temperature linear Debye-Hückel (DH) theory. From the physical point of view, due to strong electrostatic interactions, the guest particle of bare electric charge Q attracts oppositely charged electrolyte particles (counterions) to its immediate vicinity, and this decorated object may be considered as a new entity of lower renormalized charge $Q_{\rm ren}$. The idea of renormalized charge was confirmed in the framework of the nonlinear Poisson-Boltzmann (PB) approach:[3,5,6] there is a renormalized-charge prefactor, $Q_{\rm ren}$, to the usual Yukawa decay which is different from the bare charge Q of the guest particle. An interesting point is that, as the absolute value of Q increases to infinity, the renormalized charge saturates monotonically at some finite value $Q_{\rm ren}^{\rm sat}$. The possibility of a more general phenomenon of *potential* saturation was studied from a general point of view in ref. (7) and on the exactly solvable 2D Coulomb gas at the Thirring free-fermion point in ref. (8).

The rigorous validity of the PB approach is restricted, under certain conditions making the nonlinear PB theory superior to the linear DH one,[9] to a specific scaling regime of the infinite-temperature limit.[10] To go beyond this mean-field description, one has to incorporate electrostatic correlations among the electrolyte particles, like it was done, e.g., in refs. (11, 12). Such approaches always involve some plausible, but not rigorously justified, arguments and approximation schemes.

Another strategy is to concentrate on simplified models which keep the Coulomb nature of particle interactions and simultaneously admit an exact solution. Specific two-dimensional (2D) Coulomb systems with logarithmic pairwise interactions among charged constituents, where the electrolyte is modelled by an infinite symmetric Coulomb gas, belong to such category of models. The 2D Coulomb gas of $\pm$ unit pointlike charges is stable against the thermodynamic collapse of positive-negative pairs of charges at high enough temperatures, namely for $\beta < 2$ with β being the (dimensionless) inverse temperature. In this stability region, the equilibrium statistical mechanics of the Coulomb gas is exactly solvable via an equivalence with the integrable (1+1)-dimensional sine-Gordon theory; for a short review, see ref. (13). An extension of the exact treatment of the stable 2D Coulomb gas to the presence of some *pointlike* guest charge(s) was done in ref. (14). The problem of one (two) guest charge(s) immersed in the 2D Coulomb plasma was shown to be related to the evaluation of one-point (two-point) expectation values of the exponential field in the equivalent sine-Gordon model. Based on recent progress in the latter topic, two main results were obtained. Firstly, an explicit formula for the chemical potential of single guest charge Q was found in the guest-charge stability (no collapse of Q with a unit plasma counterion) region $\beta|Q| < 2$. Secondly, the asymptotic large-distance behavior of an effective interaction between two guest charges Q and Q' was derived. As a by-product of this result, considering that Q' corresponds to either $+1$ or -1 charged Coulomb-gas species, the concept of renormalized charge was confirmed and the explicit dependence of Q_{ren} on β and Q, valid rigorously in the whole stability range of $|Q| < 2/\beta$, was established. For a fixed $\beta < 2$, the renormalized charge Q_{ren}, considered as a function of, say positive, bare charge Q, exhibits a maximum at $Q = 2/\beta - 1/2$: this non-monotonic behavior resembles the one observed in the Monte Carlo[11] and molecular-dynamics[15] simulations of the salt-free (only counterions are present) colloidal Wigner-Seitz cell model. At the stability border $Q = 2/\beta$, Q_{ren} attains a finite value. This fact was an inspiration for a conjecture, referred to as "regularization hypothesis", about the possibility of an analytic continuation of the formula for Q_{ren} from the stability region to the collapse region $Q \geq 2/\beta$[14].

The validity of the regularization hypothesis was put in doubts by Téllez[16] who calculated in detail the formula for the renormalized charge within the 2D nonlinear PB theory. The PB approach describes correctly, under certain conditions [10], the scaling regime of limits $\beta \to 0$ and $Q \to \infty$ with the product βQ being finite. As was expected, in the guest-charge stability region $\beta|Q| < 2$, the exact formula for Q_{ren},[14] taken with $\beta \to 0$, was reproduced. When $\beta|Q| \geq 2$, a hard core of radius $\sigma > 0$, impenetrable for electrolyte particles, has to be attached to the guest charge Q in order to prevent its collapse with electrolyte counterions. For the determination of the renormalized charge in this regularized case, the connexion problem of the PB equation to relate the large-distance behavior of the induced electric potential with its short-distance expansion[17] is of primary importance. The numerical PB results of ref. (16) show that Q_{ren} is always an increasing function of the bare Q, and saturates at a finite value as $Q \to \infty$. In particular, when the dimensionless positive hard-core radius $\hat{\sigma} = \kappa\sigma$ (κ denotes the inverse Debye length) is very small, $\hat{\sigma} \to 0$, the renormalized charge saturates at the value given by

$$\beta Q_{\text{ren}}^{(\text{sat})} = \frac{4}{\pi} \quad \text{for } all \text{ values of } \beta Q \geq 2. \tag{1.1}$$

This is a manifestation of the Manning-Oosawa counterion condensation[18,19] known in the theory of cylindrical polyelectrolytes.

The study of the renormalization charge within the nonlinear PB approach[16] is trustworthy. However, there are two open problems. Firstly, the rigorous validity of the PB theory was proved for 3D electrolytes in the presence of some *continuous* external charge distributions.[10] The hard-core interaction between the external guest charge and the Coulomb-gas particles, which is so relevant in the questionable guest-charge collapse region $\beta|Q| \geq 2$, was not considered in Kennedy's proof. The second problem, which is probably even more important, is related to the limit $\beta \to 0$ considered in the PB approach. This limit represents a very strong restriction which prevents one from seeing nontrivialities in the plot of $Q_{\rm ren}$ versus Q appearing at a strictly finite (nonzero) β, like the previously mentioned existence of a maximum at the point $Q = 2/\beta - 1/2$.[14] It is evident that the monotonic increase of $\beta Q_{\rm ren}$ as the function of βQ to its saturation value, predicted by the PB theory, is certainly not true for a finite inverse temperature β.

The aim of the present paper is to extend the exact renormalized-charge treatment of ref. (14) to the guest charge with a hard core of radius σ, which allows us to go beyond the stability border $\beta|Q| = 2$ of the pointlike guest charge. By its spirit, the applied method is similar to the one used in ref. (20) to include hard cores around charged species of the infinite 2D Coulomb gas itself. In the limit $\hat{\sigma} \to 0$, i.e., when the hard-core radius of the guest particle is much smaller than the mean interparticle distance of the Coulomb-gas species, due to the counterion condensation in the extended region $\beta|Q| > 2$, the renormalized charge $Q_{\rm ren}$ turns out to be a periodic function of the bare charge Q with period 1. The renormalized charge therefore does not saturate at a specific finite value as $|Q| \to \infty$, but oscillates between two extreme values. Such behavior indicates that, for a strictly nonzero β, the Manning-Oosawa counterion condensation phenomenon[18,19] should be revisited. In the scaling PB regime, one recovers correctly the standard results including the saturation formula (1.1). The obtained results disprove the regularization hypothesis proposed in the previous work.[14]

The paper is organized as follows. Section 2 reviews the known exact information about the equilibrium statistical mechanics of the infinite 2D Coulomb gas, including the complete thermodynamics and both short- and large-distance asymptotic behaviors of two-point correlation functions. In this section, we introduce the notation and present important formulas which are extensively used throughout the whole paper. Section 3 deals with the chemical potential of the guest charge immersed in the 2D Coulomb gas. The case of the pointlike guest charge is analyzed in Section 3.1, the inclusion of a hard core around the guest charge is the subject of Section 3.2. The renormalization of the guest charge is studied in Section 4. In Section 4.1, the renormalization of the pointlike guest charge is briefly reviewed following the derivation of ref. (14). In Section 4.2, relevant $\hat{\sigma}$-corrections due to the presence of the hard core are systematically generated in the formula for the renormalized charge. A recapitulation is given in Section 5.

2. BULK PROPERTIES OF THE 2D COULOMB GAS

2.1. Integrability

We consider a classical, i.e. non quantum, Coulomb gas consisting of two species of pointlike particles with opposite unit charges $q \in \{+1, -1\}$ (for simplicity, the elementary charge e is put to 1), constrained to an infinite 2D plane

Λ of points $\mathbf{r} \in R^2$. The interaction energy of a set of particles $\{q_j, \mathbf{r}_j\}$ is given by $\sum_{j<k} q_j q_k v(|\mathbf{r}_j - \mathbf{r}_k|)$, where the Coulomb potential $v(\mathbf{r}) = -\ln(|\mathbf{r}|/r_0)$ (the free length constant r_0 will be set to unity for simplicity) is the regular solution of the 2D Poisson equation $\Delta v(\mathbf{r}) = -2\pi\delta(\mathbf{r})$. The equilibrium statistical mechanics of the system is usually treated in the grand canonical ensemble, characterized by the dimensionless inverse temperature β and by the couple of particle fugacities z_+ and z_-. Since the length scale r_0 was set to unity, the true dimension of $z_\pm$ is $[\text{length}]^{-2+(\beta/2)}$. The bulk Coulomb gas is neutral, and thus its thermodynamic properties depend only on the combination $\sqrt{z_+ z_-}$.[21] It is therefore possible to set $z_+ = z_- = z$; however, at some places, in order to distinguish between the $+$ and $-$ charges, we shall keep the notation $z_\pm$. The system of pointlike particles is stable against the UV collapse of positive-negative pairs of unit charges provided that the corresponding Boltzmann weight $\exp[\beta v(\mathbf{r})] = |\mathbf{r}|^{-\beta}$ can be integrated at short distances in 2D, i.e., at high enough temperatures such that $\beta < 2$. In what follows, we shall restrict ourselves to this stability range of inverse temperatures.

The grand partition function Ξ of the 2D Coulomb gas can be turned via the Hubbard-Stratonovich transformation (see, e.g., ref. (22)) into

$$\Xi(z) = \frac{\int \mathcal{D}\phi \exp[-S(z)]}{\int \mathcal{D}\phi \exp[-S(0)]} \tag{2.1}$$

with

$$S(z) = \int_\Lambda \mathrm{d}^2 r \left[\frac{1}{16\pi} (\nabla\phi)^2 - 2z\cos(b\phi) \right], \quad b = \sqrt{\frac{\beta}{4}} \tag{2.2}$$

being the Euclidean action of the $(1+1)$-dimensional sine-Gordon model. Here, $\phi(\mathbf{r})$ is a real scalar field and $\int \mathcal{D}\phi$ denotes the functional integration over this field. The sine-Gordon coupling constant b depends only on the inverse temperature β of the Coulomb gas. The fugacity z is renormalized by the (diverging) self-energy term $\exp[\beta v(\mathbf{0})/2]$ which disappears from statistical relations under the conformal short-distance normalization of the exponential fields

$$\langle e^{ib\phi(\mathbf{r})} e^{-ib\phi(\mathbf{r}')} \rangle \sim |\mathbf{r} - \mathbf{r}'|^{-4b^2} \quad \text{as } |\mathbf{r} - \mathbf{r}'| \to 0. \tag{2.3}$$

For $b^2 < 1$ ($\beta < 4$), the discrete symmetry $\phi \to \phi + 2\pi n/b$ (n being an integer) of the sine-Gordon action (2.2) is spontaneously broken and therefore the sine-Gordon model is massive.[23] Its particle spectrum consists of one soliton-antisoliton pair $(S, \bar{S})$ with equal masses M, which coexist in pairs, and of $S - \bar{S}$ bound states, called "breathers" $\{B_j; j = 1, 2, \ldots < 1/\xi\}$, whose quantized number at a given b^2 depends on the inverse of the parameter

$$\xi = \frac{b^2}{1 - b^2} \quad \left(= \frac{\beta}{4 - \beta} \right). \tag{2.4}$$

The mass of the B_j breather is given by

$$m_j = 2M \sin\left(\frac{\pi\xi}{2} j\right) \tag{2.5}$$

and this breather disappears from the sine-Gordon particle spectrum just when $m_j = 2M$, i.e., $\xi = 1/j$. Note that the lightest B_1 breather is present in the spectrum up to the collapse point $b^2 = 1/2$ ($\beta = 2$).

The 2D sine-Gordon model is an integrable field theory, so that any multi-particle scattering S-matrix factorizes into a product of explicitly available two-particle S-matrices satisfying the Yang-Baxter identity.[23] Basic characteristics of the underlying theory were derived quite recently by using the method of Thermodynamic Bethe ansatz. In particular, the (dimensionless) specific grand potential ω, defined by

$$-\omega = \lim_{|\Lambda|\to\infty} \frac{1}{|\Lambda|} \ln \Xi(z), \tag{2.6}$$

was found in ref. (25):

$$-\omega = \frac{m_1^2}{8\sin(\pi\xi)}. \tag{2.7}$$

Under the conformal normalization of the exponential fields (2.3), the relationship between the fugacity z and the soliton/antisoliton mass M reads[26]

$$z = \frac{\Gamma(b^2)}{\pi\Gamma(1-b^2)} \left[M \frac{\sqrt{\pi}\,\Gamma((1+\xi)/2)}{2\Gamma(\xi/2)} \right]^{2(1-b^2)}, \tag{2.8}$$

where Γ stands for the Gamma function.

2.2. One-point Densities

The standard definition of the homogeneous number density of the Coulomb-gas species of one sign $q = \pm 1$, $n_q(\mathbf{r}) \equiv n_q$ with $\mathbf{r} \in R^2$, is the thermal average $n_q = \langle \sum_j \delta_{q,q_j} \delta(\mathbf{r} - \mathbf{r}_j) \rangle_\beta$. The charge neutrality of the system implies that $n_+ = n_- = n/2$, where n denotes the total number density of particles. The species densities are expressible as field averages over the sine-Gordon action (2.2) as follows

$$\begin{aligned} n_q &= z_q \langle e^{iqb\phi} \rangle \quad (q = \pm 1) \\ &= \frac{z}{2} \frac{\partial(-\omega)}{\partial z}. \end{aligned} \tag{2.9}$$

This equality and the relation (2.7) determine explicitly the density-fugacity relationship:[27]

$$\frac{n^{1-(\beta/4)}}{z} = 2\left(\frac{\pi\beta}{8}\right)^{\beta/4} \frac{\Gamma(1-(\beta/4))}{\Gamma(1+(\beta/4))} \left[F\left(\frac{1}{2}, \frac{\beta}{4-\beta}; 1+\frac{\beta}{2(4-\beta)}; 1\right)\right]^{1-(\beta/4)}, \tag{2.10}$$

where $F \equiv {}_2F_1$ is the hypergeometric function which can be expressed in terms of the tangent and Gamma functions. Based on this density-fugacity relationship, the complete thermodynamics of the 2D Coulomb gas was derived in the whole stability regime of pointlike charges $\beta < 2$.[27]

The excess (i.e., over ideal) chemical potential of the Coulomb-gas species $q = \pm 1$, μ_q^{ex}, is given by

$$\exp\left(-\beta\mu_q^{\text{ex}}\right) = \frac{n_q}{z_q} = \langle \mathrm{e}^{\mathrm{i}qb\phi} \rangle, \quad q = \pm 1. \tag{2.11}$$

Let μ_Q^{ex} with arbitrarily valued real Q represents an extended definition of the excess chemical potential: μ_Q^{ex} is the reversible work which has to be done in order to bring a pointlike guest particle of charge Q from infinity into the bulk interior of the considered Coulomb gas. It was shown in ref. 14 that μ_Q^{ex} is expressible in the sine-Gordon format as follows

$$\exp\left(-\beta\mu_Q^{\text{ex}}\right) = \langle \mathrm{e}^{\mathrm{i}Qb\phi} \rangle. \tag{2.12}$$

When $Q = \pm 1$, one recovers the previous result (2.11) valid for the Coulomb-gas constituents. Due to the obvious symmetry relation $\langle \mathrm{e}^{\mathrm{i}a\phi} \rangle = \langle \mathrm{e}^{-\mathrm{i}a\phi} \rangle$ valid for any real-valued parameter a, it holds that $\mu_Q^{\text{ex}} = \mu_{-Q}^{\text{ex}}$.

A general formula for the expectation value of the exponential field $\langle \mathrm{e}^{\mathrm{i}a\phi} \rangle$ was conjectured by Lukyanov and Zamolodchikov.[24] In the notation of Eq. (2.12), $a = Qb$, their formula reads

$$\langle \mathrm{e}^{\mathrm{i}Qb\phi} \rangle = \left[\frac{\pi z \Gamma(1-b^2)}{\Gamma(b^2)}\right]^{(Qb)^2/(1-b^2)} \exp\left[I_b(Q)\right], \quad |Q| < \frac{1}{2b^2} \tag{2.13}$$

with

$$I_b(Q) = \int_0^\infty \frac{\mathrm{d}t}{t}\left[\frac{\sinh^2(2Qb^2t)}{2\sinh(b^2t)\sinh(t)\cosh[(1-b^2)t]} - 2Q^2b^2\mathrm{e}^{-2t}\right]. \tag{2.14}$$

The integral (2.14) is finite provided that $|Q| < 1/(2b^2)$: at $|Q| = 1/(2b^2)$ the integrated function behaves like $1/t$ for $t \to \infty$ what causes the logarithmic divergence. In the Coulomb-gas format, the interaction Boltzmann factor of the guest Q charge with an opposite unit plasma counterion at distance r, $r^{-\beta|Q|}$, is integrable at small 2D distances r if and only if $\beta|Q| < 2$. In terms of the sine-Gordon coupling constant $b^2 = \beta/4$, the stability region for μ_Q^{ex} is therefore expected to be $|Q| < 1/(2b^2)$ and the couple of Eqs. (2.13) and (2.14) passes the collapse test.

2.3. Two-point Densities

At the two-particle statistical level, one introduces the two-body densities $n^{(2)}_{qq'}(\mathbf{r}, \mathbf{r}') = \langle \sum_{j \neq k} \delta_{q,q_j} \delta(\mathbf{r} - \mathbf{r}_j) \delta_{q',q_k} \delta(\mathbf{r}' - \mathbf{r}_k) \rangle_\beta$ which are translationally invariant in the infinite 2D space, $n^{(2)}_{qq'}(\mathbf{r}, \mathbf{r}') = n^{(2)}_{qq'}(|\mathbf{r} - \mathbf{r}'|)$. The two-body density is expressible as an average over the sine-Gordon action (2.2) as follows

$$n^{(2)}_{qq'}(r) = z_q z_{q'} \langle \mathrm{e}^{\mathrm{i}qb\phi(\mathbf{0})} \mathrm{e}^{\mathrm{i}q'b\phi(\mathbf{r})} \rangle; \quad q, q' = \pm 1. \tag{2.15}$$

In close analogy with the previous case of one-body densities, it was shown in ref. (14) that the effective interaction energy of two guest charges immersed in the 2D Coulomb gas is expressible in terms of more general two-point correlation functions of exponential fields: $\langle \mathrm{e}^{\mathrm{i}Qb\phi(\mathbf{0})} \mathrm{e}^{\mathrm{i}Q'b\phi(\mathbf{r})} \rangle$ with real-valued charge parameters Q and Q'. A systematic generation of the short- and large-distance asymptotic expansions for these two-point correlation functions is available with the aid of special field-theoretical methods.

The short-distance expansion of $\langle \mathrm{e}^{\mathrm{i}Qb\phi(\mathbf{0})} \mathrm{e}^{\mathrm{i}Q'b\phi(\mathbf{r})} \rangle$ can be obtained by using the method of Operator Product Expansion (OPE).[28] The OPE has the form[29,30]

$$\mathrm{e}^{\mathrm{i}Qb\phi(\mathbf{0})} \mathrm{e}^{\mathrm{i}Q'b\phi(\mathbf{r})} = \sum_{n=-\infty}^{\infty} \left\{ C^{n,0}_{QQ'}(r) \mathrm{e}^{\mathrm{i}(Q+Q'+n)b\phi}(\mathbf{0}) + \cdots \right\}, \tag{2.16}$$

where the dots stand for subleading contributions of the descendants of $\mathrm{e}^{\mathrm{i}(Q+Q'+n)b\phi}$, like $(\partial\phi)^2(\bar{\partial}\phi)^2 \mathrm{e}^{\mathrm{i}(Q+Q'+n)b\phi}$, etc. The coefficients C read

$$C^{n,0}_{QQ'}(r) = z^{|n|} r^{4b^2[QQ'+n(Q+Q')+n^2/2]+2|n|(1-b^2)} f^{n,0}_{QQ'}\left(z^2 r^{4-4b^2}\right), \tag{2.17}$$

where the functions f admit analytic series expansions

$$f^{n,0}_{QQ'}(t) = \sum_{j=0}^{\infty} f^{n,0}_j(Q, Q') t^j. \tag{2.18}$$

Each coefficient $f^{n,0}_j$ is expressible as a $2(|n| + j)$-fold Coulomb integral defined in the infinite plane R^2. The leading terms $f^{n,0}_0(Q, Q')$ in the series (2.18) are expressible as

$$\begin{aligned} f^{0,0}_0(Q, Q') &= 1, \\ f^{n,0}_0(Q, Q') &= j_n(Qb^2, Q'b^2, b^2) \quad \text{for } n > 0, \\ f^{n,0}_0(Q, Q') &= j_{|n|}(-Qb^2, -Q'b^2, b^2) \quad \text{for } n < 0. \end{aligned} \tag{2.19}$$

Here,

$$j_n(a,a',b^2) = \frac{1}{n!}\int \prod_{j=1}^{n}(\mathrm{d}^2 r_j |\mathbf{r}_j|^{4a}|\mathbf{1}-\mathbf{r}_j|^{4a'}) \prod_{j<k} |\mathbf{r}_j - \mathbf{r}_k|^{4b^2} \tag{2.20}$$

with **1** being a point on the unit circle, say (1, 0). This integral is convergent if and only if the parameters (a, a', b^2) fulfill the inequalities $a > -1/2$, $a' > -1/2$ and $a + a' < -(n-1)b^2 - 1/2$. The integral (2.20) was evaluated by Dotsenko and Fateev:[31,32]

$$j_n(a,a',b^2) = \left[\frac{\pi}{\gamma(b^2)}\right]^n \prod_{j=1}^{n}\gamma(jb^2)\prod_{k=0}^{n-1}\gamma(1+2a+kb^2)\gamma(1+2a'+kb^2)$$
$$\times\gamma\left(-1-2a-2a'-(n-1+k)b^2\right). \tag{2.21}$$

Hereinafter, we use the notation $\gamma(t) = \Gamma(t)/\Gamma(1-t)$. The result (2.21) represents an analytic continuation of the integral (2.20) to all values of the parameters (a, a', b^2). We would like to emphasize that the OPE algebra (2.16) is the operation which can be used in any multi-point correlation function of exponential fields to reduce its order as soon as a couple of points is close to one another.

The large-distance asymptotic expansion of the truncated two-point correlation functions

$$\langle \mathrm{e}^{\mathrm{i}Qb\phi(\mathbf{0})}\mathrm{e}^{\mathrm{i}Q'b\phi(\mathbf{r})}\rangle^{\mathrm{T}} = \langle \mathrm{e}^{\mathrm{i}Qb\phi(\mathbf{0})}\mathrm{e}^{\mathrm{i}Q'b\phi(\mathbf{r})}\rangle - \langle \mathrm{e}^{\mathrm{i}Qb\phi}\rangle\langle \mathrm{e}^{\mathrm{i}Q'b\phi}\rangle \tag{2.22}$$

can be obtained by using the form-factor method.[33] The form-factor representation is formally expressed as an infinite convergent series over multi-particle intermediate states,

$$\langle \mathrm{e}^{\mathrm{i}Qb\phi(\mathbf{0})}\mathrm{e}^{\mathrm{i}Q'b\phi(\mathbf{r})}\rangle^{\mathrm{T}} = \sum_{N=1}^{\infty}\frac{1}{N!}\sum_{\epsilon_1,\ldots,\epsilon_N}\int_{-\infty}^{\infty}\frac{\mathrm{d}\theta_1\ldots\mathrm{d}\theta_N}{(2\pi)^N}F_Q(\theta_1,\ldots,\theta_N)_{\epsilon_1,\ldots,\epsilon_N}$$
$$\times\,{}^{\epsilon_N,\ldots,\epsilon_1}F_{Q'}(\theta_N,\ldots,\theta_1)\mathrm{e}^{-r\sum_{j=1}^{N} m_{\epsilon_j}\cosh\theta_j}. \tag{2.23}$$

Here, ϵ indexes the particles of the sine-Gordon spectrum (say $\epsilon = +/-$ for a soliton/antisoliton and $\epsilon = j$ for the B_j breather), the rapidity $\theta \in (-\infty, \infty)$ parameterizes the energy and the momentum of the particles, and F denotes the corresponding form factors. In the large-distance limit $r \to \infty$, the dominant term on the rhs of Eq. (2.23) corresponds to the intermediate state with the minimum value of the total particle mass $\sum_{j=1}^{N} m_{\epsilon_j}$, at the point of vanishing rapidities. In the stability region of the pointlike Coulomb gas $0 \le b^2 < 1/2$, the lightest particle, which can exist in the spectrum alone, is the B_1 breather of mass m_1. For this particle, the one-particle form factors $F_Q(\theta)_1$ and ${}^1F_{Q'}(\theta) = F_{Q'}(\theta)_1$ were calculated in refs. (34) and (35):

$$F_Q(\theta)_1 = -\mathrm{i}\langle \mathrm{e}^{\mathrm{i}Qb\phi}\rangle \sqrt{\pi\lambda}\frac{\sin(\pi\xi Q)}{\sin(\pi\xi)}, \tag{2.24}$$

where

$$\lambda = \frac{4}{\pi}\sin(\pi\xi)\cos\left(\frac{\pi\xi}{2}\right)\exp\left(-\int_0^{\pi\xi}\frac{\mathrm{d}t}{\pi}\frac{t}{\sin t}\right) \tag{2.25}$$

and ξ is defined in Eq. (2.4). Since the form factor (2.24) does not depend on the rapidity, the integration over θ in (2.23) can be done explicitly by using the relation

$$\int_{-\infty}^{\infty}\frac{\mathrm{d}\theta}{2}\mathrm{e}^{-rm_1\cosh\theta} = K_0(m_1 r) \underset{r\to\infty}{\sim} \left(\frac{\pi}{2m_1 r}\right)^{1/2}\mathrm{e}^{-m_1 r}. \tag{2.26}$$

Here, K_0 is the modified Bessel function of second kind. Consequently,

$$\frac{\langle \mathrm{e}^{\mathrm{i}Qb\phi(\mathbf{0})}\mathrm{e}^{\mathrm{i}Q'b\phi(\mathbf{r})}\rangle^{\mathrm{T}}}{\langle \mathrm{e}^{\mathrm{i}Qb\phi}\rangle\langle \mathrm{e}^{\mathrm{i}Q'b\phi}\rangle} \underset{r\to\infty}{\sim} -[Q][Q']\lambda\left(\frac{\pi}{2m_1 r}\right)^{1/2}\mathrm{e}^{-m_1 r}, \tag{2.27}$$

where the symbol $[Q]$ stands for the ratio

$$[Q] = \frac{\sin(\pi\xi Q)}{\sin(\pi\xi)}. \tag{2.28}$$

We see that, at large distance r, the truncated two-point correlation function factorizes into the product of separate phase contributions $[Q]$ and $[Q']$. The inverse correlation length in the exponential decay, m_1, is determined exclusively by the Coulomb-gas system. Using Eqs. (2.5)–(2.10), m_1 is expressible as

$$m_1 = \kappa\left[\frac{\sin(\pi\beta/(4-\beta))}{\pi\beta/(4-\beta)}\right]^{1/2}. \tag{2.29}$$

Here, $\kappa = \sqrt{2\pi\beta n}$ denotes the inverse Debye length; in the high-temperature $\beta \to 0$ limit one has $m_1 \sim \kappa$. The formula (2.29) describes the renormalization of the inverse correlation length at a finite temperature.

3. CHEMICAL POTENTIAL OF THE GUEST CHARGE

3.1. Pointlike Guest Charge

We consider a pointlike, without any loss of generality say positive, guest charge $Q > 0$ immersed in the bulk of the 2D Coulomb gas. Since the elementary charge was chosen as $e = 1$, Q is the valence and as such it should be an integer. However, if the plasma is composed of opposite multivalent $\pm q$ $(q = 2, 3, \ldots)$ charges, one can still represent this plasma by the Coulomb gas of unit ± 1 charges with the coupling constant βq^2 while the guest charge should be rescaled as Q/q. The most general situation is therefore described by the Coulomb gas of ± 1 charges and by the rational value of Q. The chemical potential of the guest charge is given by the relation (2.12), complemented by the explicit formulas (2.13) and (2.14) for the expectation values of exponential fields in the sine-Gordon model. Equations (2.13) and (2.14) are valid provided that $Q < Q_c$, where

$$Q_c = \frac{1}{2b^2} \quad \left(= \frac{2}{\beta} \right) \tag{3.1}$$

is the "collapse" value of the guest charge. Note that $Q_c > 1$ in the considered stability regime of the Coulomb gas $\beta < 2$ ($b^2 < 1/2$).

Equations (2.13) and (2.14) can be continued analytically to the region $Q > Q_c$ by using the reflection method developed in refs. (36, 37). Exploring a close relationship between the Liouville and sine-Gordon theories, it was argued that the sine-Gordon expectation values $\langle e^{ia\phi} \rangle$ obey the relation

$$\langle e^{ia\phi} \rangle = R(a) \langle e^{-i(a+\mathcal{G})\phi} \rangle, \tag{3.2}$$

where $\mathcal{G} = b^{-1} - b$, and the reflection amplitude reads

$$R(a) = \left[\frac{\pi z \Gamma(1-b^2)}{b^2 \Gamma(1+b^2)} \right]^{-(2a+\mathcal{G})/b} \frac{\Gamma\left(1 + \frac{2a}{b} + \frac{\mathcal{G}}{b}\right) \Gamma(1 - 2ab - \mathcal{G}b)}{\Gamma\left(1 - \frac{2a}{b} - \frac{\mathcal{G}}{b}\right) \Gamma(1 + 2ab + \mathcal{G}b)}. \tag{3.3}$$

Equations (2.13) and (2.14), in the notation of $a = Qb$ restricted to $|a| < 1/(2b)$, represent just the "minimal" solution of the reflection relations (3.2) and (3.3). Using successively the reflection formula (3.2), any expectation value $\langle e^{ia\phi} \rangle$ with $|a| > 1/(2b)$ can be reduced to a product of amplitudes of type (3.3) and the expectation value of the exponential field with a phase whose absolute value $< 1/(2b)$.

For our purpose, we take advantage of the symmetry property $\langle e^{ia\phi} \rangle = \langle e^{-ia\phi} \rangle$ and rewrite the reflection relation (3.2), taken with $a = Qb$, in the following way

$$\begin{aligned} \langle e^{iQb\phi} \rangle &= \langle e^{-iQb\phi} \rangle \\ &= R(-Qb) \langle e^{i(Qb-\mathcal{G})\phi} \rangle, \end{aligned} \tag{3.4}$$

where

$$R(-Qb) = \left[\frac{\pi z \Gamma(1-b^2)}{b^2 \Gamma(1+b^2)} \right]^{2Q+1-b^{-2}} \frac{\Gamma(-2Q + b^{-2}) \Gamma(2Qb^2 + b^2)}{\Gamma(2 + 2Q - b^{-2}) \Gamma(2 - 2Qb^2 - b^2)}. \tag{3.5}$$

While the minimal relations (2.13) and (2.14) are applicable to $-Q_c < Q < Q_c$, the "first-order" relations (3.4) and (3.5) hold for $-Q_c < Q - (\mathcal{G}/b) < Q_c$, which is equivalent to

$$Q_c - 1 < Q < Q_c + (b^{-2} - 1). \tag{3.6}$$

In the considered stability region $b^2 < 1/2$, one has $Q_c > 1$, so that the lower bound for the guest charge Q in Eq. (3.6) is a positive number, and $b^{-2} - 1 > 1$, so that the upper bound for Q in Eq. (3.6) is larger than $Q_c + 1$.

The only singularities (in particular, simple poles) of the reflection amplitude $R(-Qb)$ in Eq. (3.5) come from the Gamma function $\Gamma(-2Q + b^{-2})$, when its argument attains zero or a negative integer, i.e., at the infinite sequence of equidistant Q-points

$$Q_n^* = Q_c + \frac{n}{2}; \quad n = 0, 1, 2, \ldots. \tag{3.7}$$

The first three singular Q-points, which will be of special interest in what follows, explicitly read

$$Q_0^* = Q_c, \quad Q_1^* = Q_c + \frac{1}{2}, \quad Q_2^* = Q_c + 1; \tag{3.8}$$

note that these points lie in the range (3.6) of the applicability of the first-order reflection relations (3.4) and (3.5). Using the residuum formula for the Gamma function

$$\lim_{\epsilon \to 0^+} \Gamma(-n + \epsilon) = \frac{(-1)^n}{n!} \frac{1}{\epsilon} \quad (n = 0, 1, 2, \ldots), \tag{3.9}$$

Eqs. (3.4) and (3.5) imply, as $\epsilon \to 0^+$, the following singular behaviors

$$\langle e^{iQb\phi} \rangle \sim \frac{\pi z}{2b^2 \epsilon} \left\langle e^{i(Q_0^* - 1)b\phi} \right\rangle \quad \text{for } Q = Q_0^* - \epsilon, \tag{3.10}$$

$$\langle e^{iQb\phi} \rangle \sim -\frac{1}{4\epsilon} \left[\frac{\pi z \Gamma(1 - b^2)}{b^2 \Gamma(1 + b^2)} \right]^2 \frac{\Gamma(1 + 2b^2)}{\Gamma(1 - 2b^2)} \times \left\langle e^{i(Q_1^* - 2)b\phi} \right\rangle \quad \text{for } Q = Q_1^* - \epsilon, \tag{3.11}$$

and so on.

3.2. Guest Charge with Hard Core

Let us introduce an impenetrable hard disc of radius σ around the positively charged $Q > 0$ guest particle localized in the bulk interior of the 2D Coulomb gas, say at the origin $\mathbf{r} = \mathbf{0}$. The radius σ has the dimension of length; the most natural dimensionless quantity involving σ is chosen as

$$\hat{\sigma} = m_1 \sigma, \tag{3.12}$$

where m_1 is the inverse correlation length of the Coulomb-gas constituents given by Eq. (2.29). The corresponding excess chemical potential of the guest charge will be denoted by $\mu_Q^{\mathrm{ex}}(\sigma)$. As soon as $\sigma > 0$, $\mu_Q^{\mathrm{ex}}(\sigma)$ must be finite for any value of Q. It was shown in ref. (14) that $\mu_Q^{\mathrm{ex}}(\sigma)$ is expressible in the sine-Gordon format as follows

$$\exp\left[-\beta\mu_Q^{\mathrm{ex}}(\sigma)\right] = \langle \mathrm{e}^{\mathrm{i}Qb\phi(\mathbf{0})}\rangle_\sigma, \tag{3.13}$$

where the average $\langle\cdots\rangle_\sigma$ is defined by

$$\langle\cdots\rangle_\sigma = \frac{1}{\Xi(z)}\int \mathcal{D}\phi\, \mathrm{e}^{-S_\sigma(z)}\cdots, \tag{3.14}$$

$$S_\sigma(z) = S(z) + z\int_{r<\sigma} \mathrm{d}^2 r\left[\mathrm{e}^{\mathrm{i}b\phi(\mathbf{r})} + \mathrm{e}^{-\mathrm{i}b\phi(\mathbf{r})}\right]. \tag{3.15}$$

Here, $S(z)$ is the usual sine-Gordon action (2.2).

The averaged quantity $\langle \mathrm{e}^{\mathrm{i}Qb\phi(\mathbf{0})}\rangle_\sigma$ can be formally expanded around $S(z)$:

$$\langle \mathrm{e}^{\mathrm{i}Qb\phi(\mathbf{0})}\rangle_\sigma = \langle \mathrm{e}^{\mathrm{i}Qb\phi}\rangle + \sum_{n=1}^{\infty}\frac{(-z)^n}{n!} I_Q^{(n)}(\sigma), \tag{3.16}$$

where

$$I_Q^{(1)}(\sigma) = \int_{r<\sigma}\mathrm{d}^2 r\left[\left\langle \mathrm{e}^{\mathrm{i}Qb\phi(\mathbf{0})}\mathrm{e}^{\mathrm{i}b\phi(\mathbf{r})}\right\rangle + \left\langle \mathrm{e}^{\mathrm{i}Qb\phi(\mathbf{0})}\mathrm{e}^{-\mathrm{i}b\phi(\mathbf{r})}\right\rangle\right], \tag{3.17}$$

$$\begin{aligned} I_Q^{(2)}(\sigma) = \int_{r<\sigma}\mathrm{d}^2 r\int_{r'<\sigma}\mathrm{d}^2 r'\Big[&\left\langle \mathrm{e}^{\mathrm{i}Qb\phi(\mathbf{0})}\mathrm{e}^{\mathrm{i}b\phi(\mathbf{r})}\mathrm{e}^{\mathrm{i}b\phi(\mathbf{r}')}\right\rangle + \left\langle \mathrm{e}^{\mathrm{i}Qb\phi(\mathbf{0})}\mathrm{e}^{\mathrm{i}b\phi(\mathbf{r})}\mathrm{e}^{-\mathrm{i}b\phi(\mathbf{r}')}\right\rangle \\ &+\left\langle \mathrm{e}^{\mathrm{i}Qb\phi(\mathbf{0})}\mathrm{e}^{-\mathrm{i}b\phi(\mathbf{r})}\mathrm{e}^{\mathrm{i}b\phi(\mathbf{r}')}\right\rangle + \left\langle \mathrm{e}^{\mathrm{i}Qb\phi(\mathbf{0})}\mathrm{e}^{-\mathrm{i}b\phi(\mathbf{r})}\mathrm{e}^{-\mathrm{i}b\phi(\mathbf{r}')}\right\rangle\Big], \end{aligned} \tag{3.18}$$

and so on. The series expansion (3.16), which is assumed to be convergent, represents the basis for a systematic generation of $\hat{\sigma}$-corrections to the pure sine-Gordon expectation value $\langle \mathrm{e}^{\mathrm{i}Qb\phi}\rangle$, in the limit $\hat{\sigma}\to 0$. To be more particular, for a given value of the guest charge Q, the rhs of Eq. (3.16) contains $\hat{\sigma}$-dependent terms which either vanish (and so they are irrelevant and omitted) or diverge (and so they are relevant and preserved) in the considered limit $\hat{\sigma}\to 0$. Since the chemical potential $\mu_Q^{\mathrm{ex}}(\sigma>0)$ in Eq. (3.13) is finite for any Q, these relevant $\hat{\sigma}$-dependent terms must cancel the singularities of $\langle \mathrm{e}^{\mathrm{i}Qb\phi}\rangle$ occurring at the points $\{Q_n^*\}_{n=0}^{\infty}$ given by Eq. (3.7). The main advantage of the expansion (3.16) is that, as Q increases from 0 to ∞, the singularities of $\langle \mathrm{e}^{\mathrm{i}Qb\phi}\rangle$ at the points $Q_0^*, Q_1^*, \ldots$ are eliminated successively via the respective integral terms $I_Q^{(1)}, I_Q^{(2)}, \ldots$, where the multi-point correlation functions are evaluated by using the short-distance OPE scheme described by Eqs. (2.16)–(2.21). We shall document this claim on lower levels in the next paragraphs.

In the stability region of the pointlike guest charge $0\le Q<Q_0^*$, one has

$$\langle \mathrm{e}^{\mathrm{i}Qb\phi(\mathbf{0})}\rangle_\sigma \sim \langle \mathrm{e}^{\mathrm{i}Qb\phi}\rangle \quad \text{as } \hat{\sigma}\to 0 \tag{3.19}$$

in the sense that all $\hat{\sigma}$-dependent terms on the rhs of Eq. (3.16) vanish in the limit $\hat{\sigma} \to 0$. This is a direct consequence of the fact that the chemical potential $\mu_Q^{\mathrm{ex}}(\sigma)$, given by Eq. (3.13), is finite even for $\hat{\sigma} = 0$ in the guest-charge stability region $|Q| < Q_0^*$.

When $Q_0^* - \epsilon \le Q < Q_1^*$ (in what follows, ϵ is a positive number going to zero, $\epsilon \to 0^+$), the first term on the rhs of Eq. (3.16) has to be taken into account. The corresponding integral $I_Q^{(1)}(\sigma)$, defined in Eq. (3.17), consists of two parts: the first term corresponds to the correlation function of the pointlike guest charge Q with one coion (unit Coulomb-gas charge of the same sign as Q), the second term corresponds to the correlation function of the Q-charge with one counterion. In the considered limit $\hat{\sigma} \to 0$, the systematic short-distance expansions of the underlying correlation functions, based on the OPE scheme (2.16)–(2.21), has to be applied. The dominant contribution comes from the correlation function of the Q charge and one counterion, in the leading short-distance order corresponding to the bare-Coulomb Boltzmann factor:

$$\left\langle \mathrm{e}^{\mathrm{i}Qb\phi(\mathbf{0})}\mathrm{e}^{-\mathrm{i}b\phi(\mathbf{r})}\right\rangle \underset{r\to 0}{\sim} \langle \mathrm{e}^{\mathrm{i}(Q-1)b\phi}\rangle r^{-4Qb^2}. \tag{3.20}$$

Consequently,

$$I_Q^{(1)}(\sigma) \sim 2\pi \frac{\langle \mathrm{e}^{\mathrm{i}(Q-1)b\phi}\rangle}{m_1^{2-4Qb^2}} \frac{\hat{\sigma}^{2-4Qb^2}}{2-4Qb^2} \quad \text{as } \hat{\sigma} \to 0. \tag{3.21}$$

After simple algebraic manipulations, Eq. (3.16) then reads

$$\langle \mathrm{e}^{\mathrm{i}Qb\phi(\mathbf{0})}\rangle_\sigma \underset{\hat{\sigma}\to 0}{\sim} \langle \mathrm{e}^{\mathrm{i}Qb\phi}\rangle - \frac{\pi z}{2b^2} \frac{\langle \mathrm{e}^{\mathrm{i}(Q-1)b\phi}\rangle}{m_1^{4b^2(Q_0^*-Q)}} \frac{\hat{\sigma}^{4b^2(Q_0^*-Q)}}{Q_0^*-Q}. \tag{3.22}$$

Close to the collapse value of the guest charge, i.e., when $Q = Q_0^* - \epsilon$, one expands

$$\frac{\hat{\sigma}^{4b^2(Q_0^*-Q)}}{Q_0^*-Q} = \frac{1}{\epsilon} + 4b^2 \ln \hat{\sigma} + O(\epsilon). \tag{3.23}$$

Inserting this expansion into Eq. (3.22), the leading singular term of order ϵ^{-1} cancels exactly with its counterpart in $\langle \mathrm{e}^{\mathrm{i}Qb\phi}\rangle$ [see Eq. (3.10)], and one ends up with the logarithmic dependence on the short-distance cutoff $\hat{\sigma}$:

$$\langle \mathrm{e}^{\mathrm{i}Q_0^*b\phi(\mathbf{0})}\rangle_\sigma \underset{\hat{\sigma}\to 0}{\sim} -2\pi z \langle \mathrm{e}^{\mathrm{i}(Q_0^*-1)b\phi}\rangle \ln \hat{\sigma}. \tag{3.24}$$

The outlined cancellation of singularities at Q_0^*, due to the presence of the hard core around the guest charge, is an important check of the consistency of the present expansion procedure. For $Q > Q_0^*$, the exponent of $\hat{\sigma}$ in Eq. (3.22) is negative as was anticipated.

When $Q_1^* - \epsilon \le Q < Q_2^*$, in the limit $\hat{\sigma} \to 0$, two additional relevant $\hat{\sigma}$-contributions arise on the rhs of Eq. (3.16). The first contribution has the origin, as in the above paragraph, in the integral $I_Q^{(1)}(\sigma)$ (3.17) as the result of the next-to-leading expansion term for the correlation function of the Q charge with one counterion:

$$\langle \mathrm{e}^{\mathrm{i}Qb\phi(\mathbf{0})}\mathrm{e}^{-\mathrm{i}b\phi(\mathbf{r})}\rangle \underset{r\to 0}{\sim} \langle \mathrm{e}^{\mathrm{i}(Q-1)b\phi}\rangle r^{-4Qb^2} + z\, j_1(-Qb^2, b^2, b^2)\langle \mathrm{e}^{\mathrm{i}(Q-2)b\phi}\rangle r^{2+4b^2-8Qb^2}. \tag{3.25}$$

This expansion was derived from the OPE scheme (2.16)–(2.21) assuming that $Q > 1$, which is indeed true for the considered range of Q values. Thus,

$$I_Q^{(1)}(\sigma) \sim 2\pi \frac{\langle \mathrm{e}^{\mathrm{i}(Q-1)b\phi}\rangle}{m_1^{2-4Qb^2}} \frac{\hat{\sigma}^{2-4Qb^2}}{2-4Qb^2} + 2\pi z\, j_1(-Qb^2, b^2, b^2) \times \frac{\langle \mathrm{e}^{\mathrm{i}(Q-2)b\phi}\rangle}{m_1^{4+4b^2-8Qb^2}} \frac{\hat{\sigma}^{4+4b^2-8Qb^2}}{4+4b^2-8Qb^2} \quad \text{as } \hat{\sigma} \to 0. \tag{3.26}$$

According to the Dotsenko-Fateev result (2.21), it holds

$$j_1(-Qb^2, b^2, b^2) = \pi\gamma(1-2Qb^2)\gamma(1+2b^2)\gamma(-1-2b^2+2Qb^2). \tag{3.27}$$

The second relevant $\hat{\sigma}$-contribution has the origin in the integral $I_Q^{(2)}(\sigma)$ (3.18), namely, in the correlation function $\langle \mathrm{e}^{\mathrm{i}Qb\phi(\mathbf{0})}\mathrm{e}^{-\mathrm{i}b\phi(\mathbf{r})}\mathrm{e}^{-\mathrm{i}b\phi(\mathbf{r}')}\rangle$ of the Q charge with two electrolyte counterions. At short distances r and r', this three-point correlation function is governed by the bare-Coulomb Boltzmann factor and behaves like

$$\langle \mathrm{e}^{\mathrm{i}Qb\phi(\mathbf{0})}\mathrm{e}^{-\mathrm{i}b\phi(\mathbf{r})}\mathrm{e}^{-\mathrm{i}b\phi(\mathbf{r}')}\rangle \underset{r,r'\to 0}{\sim} \langle \mathrm{e}^{\mathrm{i}(Q-2)b\phi}\rangle r^{-4Qb^2}(r')^{-4Qb^2}|\mathbf{r}-\mathbf{r}'|^{4b^2}. \tag{3.28}$$

Thus,

$$I_Q^{(2)}(\sigma) \sim J_Q^{(2)} \frac{\langle \mathrm{e}^{\mathrm{i}(Q-2)b\phi}\rangle}{m_1^{4+4b^2-8Qb^2}} \hat{\sigma}^{4+4b^2-8Qb^2} \quad \text{as } \hat{\sigma} \to 0, \tag{3.29}$$

where $J_Q^{(2)}$ is the Coulomb integral defined inside the unit disc:

$$J_Q^{(2)} = \int_{r<1} \mathrm{d}^2 r \int_{r'<1} \mathrm{d}^2 r'\, r^{-4Qb^2}(r')^{-4Qb^2}|\mathbf{r}-\mathbf{r}'|^{4b^2}. \tag{3.30}$$

The derivation of its series representation

$$J_Q^{(2)} = \frac{\pi^2}{1+b^2-2Qb^2} \sum_{j=0}^{\infty} \frac{1}{1-2Qb^2+j} \left[\frac{\Gamma(j-2b^2)}{j!\Gamma(-2b^2)} \right]^2 \tag{3.31}$$

is outlined in the Appendix, see the final Eq. (A.7). Inserting the formula (3.26) together with Eqs. (3.29) and (3.31) into the basic series expansion (3.16), after simple algebra one finally arrives at

$$\langle e^{iQb\phi(\mathbf{0})}\rangle_\sigma \underset{\hat{\sigma}\to 0}{\sim} \langle e^{iQb\phi}\rangle - \frac{\pi z}{2b^2}\frac{\langle e^{i(Q-1)b\phi}\rangle}{m_1^{4b^2(Q_0^*-Q)}}\frac{\hat{\sigma}^{4b^2(Q_0^*-Q)}}{Q_0^*-Q}$$
$$+ f_Q^{(2)}\frac{(\pi z)^2}{4b^2}\frac{\langle e^{i(Q-2)b\phi}\rangle}{m_1^{8b^2(Q_1^*-Q)}}\frac{\hat{\sigma}^{8b^2(Q_1^*-Q)}}{Q_1^*-Q}, \tag{3.32}$$

where

$$f_Q^{(2)} = \sum_{j=0}^{\infty}\frac{1}{1-2Qb^2+j}\left[\frac{\Gamma(j-2b^2)}{j!\Gamma(-2b^2)}\right]^2 - \frac{1}{\pi}j_1(-Qb^2,b^2,b^2). \tag{3.33}$$

In addition to the previous result (3.22), Eq. (3.32) contains the $\hat{\sigma}$-dependent term whose exponent is negative, and therefore relevant, just for $Q > Q_1^*$. This term removes the singularity of the pointlike expectation value $\langle e^{iQb\phi}\rangle$ at $Q = Q_1^*$ described by the relation (3.11). To show this fact, we first recall the formula (A.11) derived in the Appendix:

$$\sum_{j=0}^{\infty}\frac{1}{1-2Q_1^*b^2+j}\left[\frac{\Gamma(j-2b^2)}{j!\Gamma(-2b^2)}\right]^2 = \frac{1}{2\pi}j_1(-Q_1^*b^2,b^2,b^2). \tag{3.34}$$

Thus, at $Q = Q_1^*$, Eq. (3.33) yields

$$f_{Q_1^*}^{(2)} = -\frac{1}{2\pi}j_1(-Q_1^*b^2,b^2,b^2) = -\frac{1}{2}[\gamma(-b^2)]^2\gamma(1+2b^2)$$
$$= \frac{1}{b^2}\left[\frac{\Gamma(1-b^2)}{\Gamma(1+b^2)}\right]^2\frac{\Gamma(1+2b^2)}{\Gamma(1-2b^2)}. \tag{3.35}$$

Here, we have used the Dotsenko-Fateev result (3.27) and the standard relation $\Gamma(x+1) = x\Gamma(x)$ for the Gamma functions. When $Q = Q_1^* - \epsilon$, after inserting the expansion

$$\frac{\hat{\sigma}^{8b^2(Q_1^*-Q)}}{Q_1^*-Q} = \frac{1}{\epsilon} + 8b^2\ln\hat{\sigma} + O(\epsilon) \tag{3.36}$$

and the explicit form of $f_{Q_1^*}$ (3.35) into Eq. (3.32), the leading singular term of order ϵ^{-1} cancels exactly with its counterpart in $\langle e^{iQb\phi}\rangle$ [see Eq. (3.11)]. As before at the collapse value Q_0^* [see Eq. (3.24)], one ends up with a logarithmic dependence on the short-distance cutoff $\hat{\sigma}$ at the point Q_1^*.

Note that the exponents of the two $\hat{\sigma}$-dependent terms on the rhs of Eq. (3.32) satisfy the inequality $4b^2(Q_0^*-Q) < 8b^2(Q_1^*-Q)$ up to $Q = 2Q_1^* - Q_0^* = Q_2^*$. This means that, in the $\hat{\sigma}\to 0$ limit, the first $\hat{\sigma}$-dependent term with the exponent

$4b^2(Q_0^* - Q)$ is dominant in the whole interval $Q_0^* \le Q < Q_2^*$:

$$\langle e^{iQb\phi(\mathbf{0})}\rangle_\sigma \underset{\hat\sigma\to 0}{\sim} \frac{\pi z}{2b^2} \frac{\langle e^{i(Q-1)b\phi}\rangle}{m_1^{4b^2(Q_0^*-Q)}} \frac{\hat\sigma^{4b^2(Q_0^*-Q)}}{Q-Q_0^*}, \quad Q_0^* \le Q < Q_2^*. \tag{3.37}$$

Crossing the next singular point Q_2^*, this term becomes subleading and the second $\hat\sigma$-dependent term with the exponent $8b^2(Q_1^* - Q)$ takes the dominant role, it turns out that up to Q_4^*.

One can proceed along the above lines to higher-order $\hat\sigma$-contributions, with an increasing amount of algebraic laboriousness. In this paragraph, we indicate the general structure of the relevant $\hat\sigma$-contributions on the rhs of Eq. (3.16). From among correlation functions in the integral $I_Q^{(n)}(\sigma)$, the important one corresponds to the configuration of one guest charge Q surrounded by n counterions, $\langle e^{iQb\phi(\mathbf{0})} e^{-ib\phi(\mathbf{r}_1)} \ldots e^{-ib\phi(\mathbf{r}_n)}\rangle$. The leading short-distance expansion of this correlation function is governed by the bare-Coulomb Boltzmann factor:

$$\langle e^{iQb\phi(\mathbf{0})} e^{-ib\phi(\mathbf{r}_1)} \cdots e^{-ib\phi(\mathbf{r}_n)}\rangle \underset{r_1,\ldots,r_n\to 0}{\sim} \langle e^{i(Q-n)b\phi}\rangle r_1^{-4Qb^2} \cdots r_n^{-4Qb^2} \prod_{i<j} |\mathbf{r}_i - \mathbf{r}_j|^{4b^2}. \tag{3.38}$$

Rescaling the 2D spatial coordinates $\mathbf{r}_1, \ldots, \mathbf{r}_n$ by σ and regarding the equality $2n - 4nQb^2 + 2n(n-1)b^2 = 4nb^2(Q_{n-1}^* - Q)$, one obtains

$$I_Q^{(n)}(\sigma) \sim J_Q^{(n)} \frac{\langle e^{i(Q-n)b\phi}\rangle}{m_1^{4nb^2(Q_{n-1}^*-Q)}} \hat\sigma^{4nb^2(Q_{n-1}^*-Q)} \quad \text{as } \hat\sigma \to 0, \tag{3.39}$$

where $J_Q^{(n)}$ is a $2n$-fold Coulomb integral defined in the unit disc. This $\hat\sigma$-contribution has its counterparts, which possess the structure of Eq. (3.39) except of prefactors different from $J_Q^{(n)}$, in each of the lower-order integrals $I_Q^{(1)}(\sigma), \ldots, I_Q^{(n-1)}(\sigma)$ in the series (3.16) due to the existence of higher-order terms of the short-distance expansions of the correlation functions under integration. In analogy with Eq. (3.32), the summation over all $\hat\sigma$-contributions belonging to the family (3.39) can be formally represented as

$$f_Q^{(n)} \frac{(-2\pi z)^n}{n!(4nb^2)} \frac{\langle e^{i(Q-n)b\phi}\rangle}{m_1^{4nb^2(Q_{n-1}^*-Q)}} \frac{\hat\sigma^{4nb^2(Q_{n-1}^*-Q)}}{Q_{n-1}^* - Q}. \tag{3.40}$$

The term (3.40) becomes relevant for $Q > Q_{n-1}^*$; the prefactor $f_Q^{(n)}$ is such that it cancels exactly the singularity of the pointlike mean value $\langle e^{iQb\phi}\rangle$ at Q_{n-1}^*. Two consecutive nth and $(n+1)$st $\hat\sigma$-contributions of type (3.40) interchange their dominant role when the respective exponents $4nb^2(Q_{n-1}^* - Q)$ and $4(n+1)b^2(Q_n^* - Q)$ coincide, i.e., at the point $Q = (n+1)Q_n^* - nQ_{n-1}^* = Q_{2n}^*$.

This means that, in the $\hat{\sigma} \to 0$ limit, $\langle e^{iQb\phi(\mathbf{0})} \rangle_\sigma$ is dominated by the nth $\hat{\sigma}$-contribution (3.40) in the range $Q^*_{2(n-1)} \leq Q < Q^*_{2n}$.

4. RENORMALIZED CHARGE

4.1. Pointlike Guest Charge

The pointlike guest particle of charge $Q > 0$, localized at the origin $\mathbf{0}$ and surrounded by the infinite 2D Coulomb gas, evokes position-dependent density profiles $n_q(\mathbf{r})$ of the electrolyte species $q = \pm 1$. In the sine-Gordon field representation, these density profiles are given by[14]

$$n_q(\mathbf{r}) = z \frac{\langle e^{iQb\phi(\mathbf{0})} e^{iqb\phi(\mathbf{r})} \rangle}{\langle e^{iQb\phi} \rangle}, \quad q = \pm 1. \tag{4.1}$$

At large distances from the guest charge, the form-factor result (2.27) implies

$$n_q(r) - n_q \sim -n_q[q][Q]\lambda \left(\frac{\pi}{2m_1 r} \right)^{1/2} \exp(-m_1 r) \quad \text{as } r \to \infty. \tag{4.2}$$

Here, $[q] = q$ for $q = \pm 1$, the function $[Q]$ (2.28) is expressible in terms of the inverse temperature β as follows

$$[Q] = \frac{\sin[\pi\beta Q/(4-\beta)]}{\sin[\pi\beta/(4-\beta)]}, \tag{4.3}$$

and the parameter λ (2.25) as follows

$$\lambda = \frac{4}{\pi} \sin\left(\frac{\pi\beta}{4-\beta} \right) \cos\left(\frac{\pi\beta}{2(4-\beta)} \right) \exp\left\{ - \int_0^{\pi\beta/(4-\beta)} \frac{dt}{\pi} \frac{t}{\sin t} \right\}. \tag{4.4}$$

Using Eq. (4.2), the induced charge density ρ of electrolyte particles, defined by $\rho(\mathbf{r}) = n_+(\mathbf{r}) - n_-(\mathbf{r})$, reads

$$\rho(r) \underset{r\to\infty}{\sim} -n[Q]\lambda \left(\frac{\pi}{2m_1 r} \right)^{1/2} \exp(-m_1 r). \tag{4.5}$$

The average electrostatic potential ψ is related to the charge-density profile through the 2D Poisson equation,

$$\Delta\psi(\mathbf{r}) = -2\pi\rho(\mathbf{r}). \tag{4.6}$$

Considering the circularly symmetric Laplacian $\Delta = r^{-1} d_r(r d_r)$, the asymptotic formula (4.5) gives

$$\beta\psi(r) \underset{r\to\infty}{\sim} [Q]\lambda \left(\frac{\kappa}{m_1} \right)^2 \left(\frac{\pi}{2m_1 r} \right)^{1/2} \exp(-m_1 r). \tag{4.7}$$

In the Debye-Hückel limit $\beta \to 0$, it holds $[Q] \sim Q$, $m_1 \sim \kappa$ and $\lambda \sim \beta$. Eq. (4.7) then reduces to

$$\beta \psi_{\rm DH}(r) \underset{r\to\infty}{\sim} \beta Q \left(\frac{\pi}{2\kappa r}\right)^{1/2} \exp(-\kappa r). \tag{4.8}$$

Eq. (4.7) is consistent with Eq. (4.8), in terms of the dimensionless combinations $m_1 r$ and κr, respectively, when one introduces the renormalized charge $Q_{\rm ren}$ as follows[14]

$$\beta Q_{\rm ren} = [Q]\lambda \left(\frac{\kappa}{m_1}\right)^2. \tag{4.9}$$

We see that the dependence of the renormalized charge $Q_{\rm ren}$ on the bare charge Q enters only via the function $[Q]$ defined by Eq. (4.3). The rigorous validity of the result (4.9) is restricted to the guest-charge stability region $0 \le Q < Q_c$. Within this range of Q values, the function $[Q]$ exhibits the maximum at $Q = Q_c - 1/2$. It can be readily checked that at the collapse value of the guest charge this function satisfies the important relation

$$[Q_c] = [Q_c - 1] \tag{4.10}$$

which was not noted in the previous work.[14] The relation (4.10) reflects the effect of the counterion condensation: the renormalized charge associated with the collapse value Q_c of the bare charge is identical to that of the bare charge $Q_c - 1$ because of the condensation of one counterion from the electrolyte onto the guest particle.

4.2. Guest Charge with Hard Core

We now consider the positive guest charge Q possessing the hard core of radius σ. The evoked density profiles $n_q(\mathbf{r})$ of the electrolyte species $q = \pm 1$ vanish inside the hard disc, $n_q(\mathbf{r}) = 0$ for $r \le \sigma$. Using the formalism of ref. (14) it can be shown that outside the hard-core region the number densities of the electrolyte species are given by the following generalization of the relation (4.1)

$$n_q(\mathbf{r}) = z \frac{\langle e^{iQb\phi(\mathbf{0})} e^{iqb\phi(\mathbf{r})}\rangle_\sigma}{\langle e^{iQb\phi(\mathbf{0})}\rangle_\sigma} \quad \text{for } r > \sigma, \tag{4.11}$$

where the average $\langle \cdots \rangle_\sigma$ is defined in Eqs. (3.14) and (3.15). The relation (4.11) can be rewritten in a more convenient form:

$$n_q(\mathbf{r}) - n_q = n_q \frac{\langle e^{iQb\phi(\mathbf{0})} e^{iqb\phi(\mathbf{r})}\rangle_\sigma - \langle e^{iQb\phi(\mathbf{0})}\rangle_\sigma \langle e^{iqb\phi}\rangle}{\langle e^{iQb\phi(\mathbf{0})}\rangle_\sigma \langle e^{iqb\phi}\rangle}, \quad r > \sigma. \tag{4.12}$$

The one-point quantity $\langle e^{iQb\phi(\mathbf{0})}\rangle_\sigma$ was analyzed in Section 3.2, the large-distance analysis of the two-point correlation functions $\langle e^{iQb\phi(\mathbf{0})} e^{iqb\phi(\mathbf{r})}\rangle_\sigma$ $(q = \pm 1)$

will proceed analogously. Expanding formally $\langle\cdots\rangle_\sigma$ around the sine-Gordon action $S(z)$, one obtains

$$\langle \mathrm{e}^{\mathrm{i}Qb\phi(\mathbf{0})}\mathrm{e}^{\mathrm{i}qb\phi(\mathbf{r})}\rangle_\sigma = \langle \mathrm{e}^{\mathrm{i}Qb\phi(\mathbf{0})}\mathrm{e}^{\mathrm{i}qb\phi(\mathbf{r})}\rangle + \sum_{n=1}^{\infty} \frac{(-z)^n}{n!} I_{Q,q}^{(n)}(r;\sigma), \tag{4.13}$$

where

$$I_{Q,q}^{(1)}(r;\sigma) = \int_{r_1<\sigma} \mathrm{d}^2 r_1 \sum_{q_1=\pm 1} \langle \mathrm{e}^{\mathrm{i}Qb\phi(\mathbf{0})}\mathrm{e}^{\mathrm{i}q_1 b\phi(\mathbf{r}_1)}\mathrm{e}^{\mathrm{i}qb\phi(\mathbf{r})}\rangle, \tag{4.14}$$

$$I_{Q,q}^{(2)}(r;\sigma) = \int_{r_1<\sigma} \mathrm{d}^2 r_1 \int_{r_2<\sigma} \mathrm{d}^2 r_2 \sum_{q_1=\pm 1}\sum_{q_2=\pm 1} \langle \mathrm{e}^{\mathrm{i}Qb\phi(\mathbf{0})}\mathrm{e}^{\mathrm{i}q_1 b\phi(\mathbf{r}_1)}\mathrm{e}^{\mathrm{i}q_2 b\phi(\mathbf{r}_2)}\mathrm{e}^{\mathrm{i}qb\phi(\mathbf{r})}\rangle, \tag{4.15}$$

and so on. As before, increasing the value of the guest charge Q from 0 to ∞, the expansion (4.13) represents the basis for the systematic generation of $\hat{\sigma}$-corrections which are relevant (i.e., have a negative exponent) in the $\hat{\sigma}\to 0$ limit.

In the stability region of the pointlike guest charge $0 \le Q < Q_0^*$, one has

$$\langle \mathrm{e}^{\mathrm{i}Qb\phi(\mathbf{0})}\mathrm{e}^{\mathrm{i}qb\phi(\mathbf{r})}\rangle_\sigma \sim \langle \mathrm{e}^{\mathrm{i}Qb\phi(\mathbf{0})}\mathrm{e}^{\mathrm{i}qb\phi(\mathbf{r})}\rangle \quad \text{as } \hat{\sigma}\to 0. \tag{4.16}$$

Inserting this relation together with the previously derived one (3.19) into Eq. (4.12) and repeating the procedure of Section 4.1, one ends up with the renormalized-charge formula (4.9).

When $Q_0^* - \epsilon \le Q < Q_1^*$, the relevant contribution to the integral $I_{Q,q}^{(1)}(r;\sigma)$ (4.14) comes from the $q_1 = -1$ counterion correlation function. Since the counterion position vector $\mathbf{r}_1$ is close to the origin $\mathbf{0}$, we can apply the short-distance OPE formula

$$\mathrm{e}^{\mathrm{i}Qb\phi(\mathbf{0})}\mathrm{e}^{-\mathrm{i}b\phi(\mathbf{r}_1)} \underset{r_1\to 0}{\sim} r_1^{-4Qb^2}\mathrm{e}^{\mathrm{i}(Q-1)b\phi(\mathbf{0})}, \tag{4.17}$$

which implies

$$I_{Q,q}^{(1)}(r;\sigma) \sim 2\pi \frac{\langle \mathrm{e}^{\mathrm{i}(Q-1)b\phi(\mathbf{0})}\mathrm{e}^{\mathrm{i}qb\phi(\mathbf{r})}\rangle}{m_1^{2-4Qb^2}} \frac{\hat{\sigma}^{2-4Qb^2}}{2-4Qb^2} \quad \text{as } \hat{\sigma}\to 0. \tag{4.18}$$

The consideration of this formula in the basic expansion (4.13) leads to

$$\langle \mathrm{e}^{\mathrm{i}Qb\phi(\mathbf{0})}\mathrm{e}^{\mathrm{i}qb\phi(\mathbf{r})}\rangle_\sigma \underset{\hat{\sigma}\to 0}{\sim} \langle \mathrm{e}^{\mathrm{i}Qb\phi(\mathbf{0})}\mathrm{e}^{\mathrm{i}qb\phi(\mathbf{r})}\rangle - \frac{\pi z}{2b^2} \frac{\langle \mathrm{e}^{\mathrm{i}(Q-1)b\phi(\mathbf{0})}\mathrm{e}^{\mathrm{i}qb\phi(\mathbf{r})}\rangle}{m_1^{4b^2(Q_0^*-Q)}} \frac{\hat{\sigma}^{4b^2(Q_0^*-Q)}}{Q_0^*-Q}. \tag{4.19}$$

Close to the collapse point, $Q = Q_0^* - \epsilon$, using the large-distance form-factor representation of two-point correlation functions (2.27) and the formula (4.10), singular terms of order ϵ^{-1} disappear from the rhs of Eq. (4.19) in the same way as it was in the case of $\langle e^{iQb\phi(\mathbf{0})}\rangle_\sigma$, see Eqs. (3.22)–(3.24). Finally, inserting (4.19) into Eq. (4.12) and using the form-factor representation (2.27) together with the result (3.22) for $\langle e^{iQb\phi(\mathbf{0})}\rangle_\sigma$, one obtains

$$n_q(r) - n_q \underset{r\to\infty}{\sim} -n_q[q]g_Q^{(1)}(\sigma)\lambda\left(\frac{\pi}{2m_1 r}\right)^{1/2}\exp(-m_1 r). \tag{4.20}$$

Here,

$$g_Q^{(1)}(\sigma) = \frac{N_Q^{(1)}(\sigma)}{D_Q^{(1)}(\sigma)} \tag{4.21}$$

with

$$N_Q^{(1)}(\sigma) \underset{\hat{\sigma}\to 0}{\sim} \langle e^{iQb\phi}\rangle[Q] - \frac{\pi z}{2b^2}\frac{\langle e^{i(Q-1)b\phi}\rangle}{m_1^{4b^2(Q_0^*-Q)}}\frac{\hat{\sigma}^{4b^2(Q_0^*-Q)}}{Q_0^*-Q}[Q-1], \tag{4.22}$$

$$D_Q^{(1)}(\sigma) \underset{\hat{\sigma}\to 0}{\sim} \langle e^{iQb\phi}\rangle - \frac{\pi z}{2b^2}\frac{\langle e^{i(Q-1)b\phi}\rangle}{m_1^{4b^2(Q_0^*-Q)}}\frac{\hat{\sigma}^{4b^2(Q_0^*-Q)}}{Q_0^*-Q}. \tag{4.23}$$

The denominator $D_Q^{(1)}(\sigma)$ is equal to the one-point average $\langle e^{iQb\phi(\mathbf{0})}\rangle_\sigma$ given by Eq. (3.22) and the numerator $N_Q^{(1)}(\sigma)$ possesses the same structure as $D_Q^{(1)}(\sigma)$ but each term is multiplied by the function $[\cdots]$ (2.28) with the argument related to the phase of the corresponding averaged exponential field; we shall see that this rule takes place also on higher levels. At the collapse point $Q = Q_0^*$, the equality (4.10) implies that $g_{Q_0^*}^{(1)}(\sigma)\sim_{\hat{\sigma}\to 0}[Q_0^*]$, and one recovers the pointlike result (4.2). This means that $Q_{\rm ren}$ is the continuous function of the bare charge Q at the collapse point Q_0^*. For $Q_0^* < Q < Q_1^*$, the term of order $\hat{\sigma}^{4b^2(Q_0^*-Q)}$ is dominant in the limit $\hat{\sigma} \to 0$ in both $D_Q^{(1)}(\sigma)$ and $N_Q^{(1)}(\sigma)$, so that $g_Q^{(1)} = [Q-1]$. The repetition of the procedure of Section 4.1, starting from Eq. (4.20), then leads to

$$\beta Q_{\rm ren} = [Q-1]\lambda\left(\frac{\kappa}{m_1}\right)^2, \quad Q_0^* \le Q < Q_1^*. \tag{4.24}$$

Comparing this relation with Eq. (4.9) (valid for $Q < Q_0^*$) we see that, in the considered range of Q values, the bare charge Q of the guest particle is effectively reduced by 1 due to the condensation of one electrolyte counterion.

In the region $Q_1^* \le Q < Q_2^*$, one recovers the formula of type (4.20) with the substitution $g_Q^{(1)}(\sigma) \to g_Q^{(2)}(\sigma)$, where $g_Q^{(2)}(\sigma)$ is again the ratio of type (4.21):

$$g_Q^{(2)}(\sigma) = \frac{N_Q^{(2)}(\sigma)}{D_Q^{(2)}(\sigma)}. \tag{4.25}$$

Here, the denominator $D_Q^{(2)}(\sigma)$ is equal to the one-point average $\langle e^{iQb\phi(\mathbf{0})}\rangle_\sigma$ given by Eq. (3.32) and, as before, the numerator

$$N_Q^{(2)}(\sigma) \underset{\hat{\sigma}\to 0}{\sim} \langle e^{iQb\phi}\rangle[Q] - \frac{\pi z}{2b^2} \frac{\langle e^{i(Q-1)b\phi}\rangle}{m_1^{4b^2(Q_0^*-Q)}} \frac{\hat{\sigma}^{4b^2(Q_0^*-Q)}}{Q_0^*-Q}[Q-1]$$
$$+ f_Q^{(2)} \frac{(\pi z)^2}{4b^2} \frac{\langle e^{i(Q-2)b\phi}\rangle}{m_1^{8b^2(Q_1^*-Q)}} \frac{\hat{\sigma}^{8b^2(Q_1^*-Q)}}{Q_1^*-Q}[Q-2] \tag{4.26}$$

arises from $D_Q^{(2)}(\sigma)$ by multiplying each term by the function $[\cdots]$ with the argument related to the phase of the corresponding averaged exponential field. In the $\hat{\sigma} \to 0$ limit, once again the term of order $\hat{\sigma}^{4b^2(Q_0^*-Q)}$ is dominant in both $D_Q^{(2)}(\sigma)$ and $N_Q^{(2)}(\sigma)$ in the considered range $Q_1^* \leq Q < Q_2^*$, so that $g_Q^{(2)} = [Q-1]$. The previously derived formula (4.24) thus applies to the whole region $Q_0^* \leq Q < Q_2^*$,

$$\beta Q_{\text{ren}} = [Q-1]\lambda\left(\frac{\kappa}{m_1}\right)^2, \quad Q_0^* \leq Q < Q_2^*. \tag{4.27}$$

It can be shown that the next term in Eq. (4.26), of order $\hat{\sigma}^{8b^2(Q_1^*-Q)}$, becomes dominant in the region $Q_2^* \leq Q < Q_4^*$. Consequently, in the limit $\hat{\sigma} \to 0$,

$$\beta Q_{\text{ren}} = [Q-2]\lambda\left(\frac{\kappa}{m_1}\right)^2, \quad Q_2^* \leq Q < Q_4^*. \tag{4.28}$$

This relation describes the simultaneous condensation of two electrolyte counterions onto the guest charge Q.

With the aid of the general analysis outlined at the end of Section 3.2, one can extend the treatment, performed above in the range $Q_0^* \leq Q < Q_2^*$, to an arbitrary interval $Q_{2(n-1)}^* \leq Q < Q_{2n}^*$ ($n = 1, 2\ldots$). For a given n, the dominant term (in the $\hat{\sigma} \to 0$ limit) in $\langle e^{iQb\phi(\mathbf{0})}\rangle_\sigma$ is of type (3.40). The corresponding factor generated by $\langle e^{i(Q-n)b\phi}\rangle$ is $[Q-n]$, which implies

$$\beta Q_{\text{ren}} = [Q-n]\lambda\left(\frac{\kappa}{m_1}\right)^2, \quad Q_{2(n-1)}^* \leq Q < Q_{2n}^* \tag{4.29}$$

for $n = 1, 2, \ldots$. The bare charge Q of the guest particle is therefore effectively reduced by n due to the condensation of n electrolyte counterions. Since $Q_{2n}^* = Q_c + n$, the formula (4.29) tells us that beyond the collapse point Q_c, i.e., when $\beta Q \geq 2$, the renormalized charge Q_{ren} is the periodic function of the bare charge Q with period 1. The minimum value of Q_{ren}, given by

$$\beta Q_{\text{ren}}^{(\text{min})} = \lambda \sin\left(\frac{2\pi}{4-\beta}\right) \frac{\pi\beta/(4-\beta)}{\sin^2[\pi\beta/(4-\beta)]}, \tag{4.30}$$

is acquired at the infinite sequence of points $Q_0^*, Q_2^*, Q_4^*, \ldots$; the maximum value of $Q_{\rm ren}$, given by

$$\beta Q_{\rm ren}^{(\max)} = \lambda \frac{\pi\beta/(4-\beta)}{\sin^2[\pi\beta/(4-\beta)]}, \tag{4.31}$$

is acquired at the infinite sequence of points $Q_1^*, Q_3^*, Q_5^*, \ldots$. It is obvious that, in the considered limit $\hat{\sigma} \to 0$, the renormalized charge does not saturate at a specific finite value as $Q \to \infty$, but oscillates between the two extremes given by Eqs. (4.30) and (4.31). Note that $Q_{\rm ren}^{(\min)}$ is positive, i.e., has the sign of the bare Q, in the considered Coulomb-gas stability region $0 \le \beta < 2$; consequently, there is no charge inversion in the system.

Like for example, the plot of $Q_{\rm ren}$ versus Q at the inverse temperature $\beta = 1$ is presented in Fig. 1. The solid line represents the true dependence given by Eqs. (4.9) and (4.29). The dashed line corresponds to the "naive" analytic continuation of the pointlike formula (4.9) into the counterion-condensation region $Q \ge 2$, based on the regularization hypothesis of ref. (14).

We now check the obtained exact results on the high-temperature PB scaling regime of limits $\beta \to 0$ and $Q \to \infty$ with the product βQ being finite. In the guest-charge stability region $0 \le \beta Q < 2$, the consideration of $m_1 \sim \kappa$, $\lambda \sim \beta$

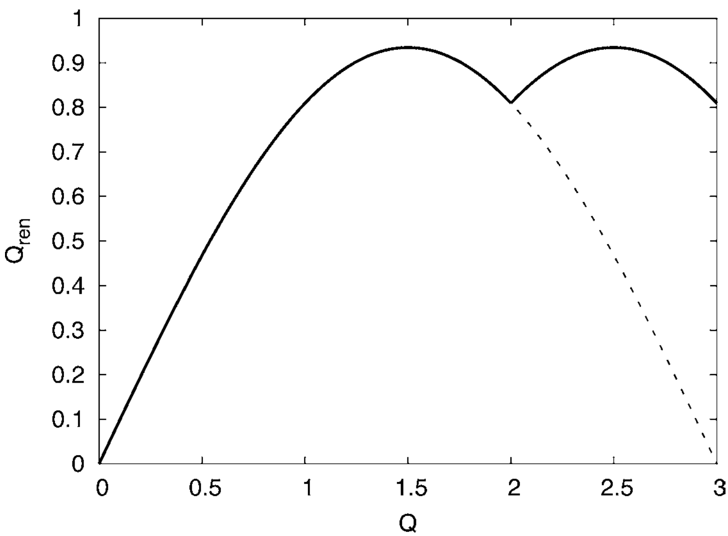

Fig. 1. $Q_{\rm ren}$ versus the bare charge Q at $\beta = 1$ (*solid line*).

and $[Q] \sim 4[\sin(\pi\beta Q/4)]/(\pi\beta)$ in Eq. (4.9) results in

$$\beta Q_{\rm ren} = \frac{4}{\pi}\sin\left(\frac{\pi\beta Q}{4}\right), \quad 0 \le \beta Q < 2. \tag{4.32}$$

Evidently, $\beta Q_{\rm ren}$ is an increasing function of βQ in the stability interval. It acquires its maximum value, equal to $4/\pi$, when βQ approaches the collapse value 2. For $\beta Q \ge 2$, the two extreme values (4.30) and (4.31) coincide in the PB scaling regime:

$$\beta Q_{\rm ren}^{(\rm min)} = \beta Q_{\rm ren}^{(\rm max)} = \frac{4}{\pi}, \quad \beta Q \ge 2. \tag{4.33}$$

This is equivalent to saying that $Q_{\rm ren}$ saturates at the value given by $\beta Q_{\rm ren}^{(\rm sat)} = 4/\pi$ for all $\beta Q \ge 2$, which is in agreement with the result (1.1) of the Manning-Oosawa theory of counterion condensation.[16] We would like to emphasize that both the monotonic increase of $\beta Q_{\rm ren}$ in the stability region $0 \le \beta Q < 2$ and the saturation of $\beta Q_{\rm ren}$ at the uniform value for $\beta Q \ge 2$ are the specific features of the $\beta \to 0$ limit. In the case of a strictly finite (nonzero) value of the inverse temperature β, the previously described non-monotonic oscillatory dependence of $Q_{\rm ren}$ on the bare charge Q takes place.

In Monte Carlo[11] and molecular-dynamics[15] simulations of the salt-free Wigner-Seitz cell model, $Q_{\rm ren}$ as the function of Q exhibits, after reaching its maximum, a slow monotonic decrease to the saturation value at $Q \to \infty$. The absence of oscillations is due to the fact that the requirement of the overall charge neutrality within one cell allows for only such absolute values of the guest charge Q which are integer multiples of the counterion (possibly multivalent) charges. In our units, this corresponds to Q being an integer. In the region $\beta Q \ge 2$ and in the limit $\hat{\sigma} \to 0$ considered in this paper, the shift of Q by 1 does not affect $Q_{\rm ren}$ what explains the absence of oscillations. The slight decrease of $Q_{\rm ren}$ to its saturation value observed in simulations is probably caused by the finite value of $\hat{\sigma}$.

5. CONCLUSION

Exact results of ref. (14), which concern thermodynamic characteristics of a pointlike guest charge Q (say, ≥ 0) immersed in the infinite 2D Coulomb gas of ± 1 charged species, are rigorously valid in the stability region of the guest particle $0 \le \beta Q < 2$. The analytic continuation of these results to the "collapse" region $\beta Q \ge 2$ turns out to be inadequate. In the region $\beta Q \ge 2$, in order to prevent the direct collapse of electrolyte counterions onto the guest charge, the hard core of radius σ, appearing in the dimensionless combination $\hat{\sigma}$ (3.12), has to be attached to the guest charge and only afterwards on can consider the limit $\hat{\sigma} \to 0$. Each quantity related to the guest particle, evaluated in the presence of the hard core, exhibits some relevant $\hat{\sigma}$-corrections with respect to its pointlike value which do not vanish, but on the contrary diverge, in the limit $\hat{\sigma} \to 0$. As Q increases, the hierarchy of relevant $\hat{\sigma}$-corrections becomes more complex and among all $\hat{\sigma}$-corrections one is dominant in the sense that it fully determines the given quantity in the limit $\hat{\sigma} \to 0$. The main technicality of this paper consists in the systematic generation of the relevant $\hat{\sigma}$-corrections as the value of Q increases and in the determination of the dominant $\hat{\sigma}$-correction.

The (excess) chemical potential of the guest charge, $\mu_Q^{\rm ex}$, was the thermodynamic quantity of interest in Section 3. In the case of the pointlike guest charge, $\mu_Q^{\rm ex}$ is expressible in terms of the expectation value of the exponential field in the 2D sine-Gordon model, see Eq. (2.12). The explicit result for this expectation value is available in the whole guest-charge stability region $0 \le \beta Q < 2$, see Eqs. (2.13) and (2.14). The nontrivial extension of this result to the region $\beta Q \ge 2$ is possible due to the existence of the reflection relations (3.4) and (3.5). These reflection relations exhibit an infinite sequence of singularities at the equidistant Q-points $\{Q_n^*\}_{n=0}^{\infty}$ given by Eq. (3.7). The chemical potential in the presence of the hard core around the guest particle is the subject of Section 3.2, where the relevant $\hat{\sigma}$-corrections to the pointlike $\mu_Q^{\rm ex}$ are systematically generated. These $\hat{\sigma}$-corrections remove the artificial singularities of the pointlike $\mu_Q^{\rm ex}$ at the points $\{Q_n^*\}_{n=0}^{\infty}$, which is an important consistency check of the generation procedure. The hierarchy of the relevant $\hat{\sigma}$-corrections becomes more and more complicated as the bare charge Q increases, however, the $\hat{\sigma}$-correction, dominant in the limit $\hat{\sigma} \to 0$, is easily detectable for arbitrary-valued Q.

The crucial Section 4 is devoted to the evaluation of the renormalized charge $Q_{\rm ren}$ of the guest particle. The case of the pointlike guest charge is briefly reviewed in Section 4.1. The inclusion of the hard core around the guest particle, studied in Section 4.2, requires the application of both short-distance (the OPE method) and large-distance (the form-factor method) asymptotic expansions of two-point correlation functions in the 2D sine-Gordon model. The systematic generation of the relevant $\hat{\sigma}$-corrections with respect to the pointlike renormalized charge is similar to the one outlined in Section 3.2 for the guest-charge chemical potential. The final result for the renormalized charge (4.29) is very simple. In the problematic region $\beta Q \ge 2$, due to the condensation of electrolyte counterions onto the guest charge, the renormalized charge $Q_{\rm ren}$ is a periodic function of the bare charge Q with period 1 and its value oscillates between the two extremes given by Eqs. (4.30) and (4.31). In the Poisson-Boltzmann scaling regime, these two extreme values coincide and one recovers the standard Manning-Oosawa type of counterion condensation with the uniform saturation value of $\beta Q_{\rm ren} = 4/\pi$ in the whole region $\beta Q \ge 2$. From this point of view, the nature of the results in the PB scaling regime is fundamentally different from that of the exact result obtained at a strictly finite value of the inverse temperature β.

APPENDIX

The integral $J_Q^{(2)}$, defined in Eq. (3.30), can be straightforwardly transformed to the form

$$J_Q^{(2)} = 4\pi \int_0^1 \mathrm{d}r\, r^{1+4b^2-4Qb^2} \int_0^r \mathrm{d}r'\, (r')^{1-4Qb^2} \times \int_0^{2\pi} \mathrm{d}\varphi \left[1 - 2\cos\varphi \left(\frac{r'}{r}\right) + \left(\frac{r'}{r}\right)^2 \right]^{2b^2}. \tag{A.1}$$

The expression in the square bracket can be expanded in the ratio (r'/r) by using Gegenbauer polynomials $\{C_j^\lambda(t)\}$, defined as the coefficients in the power-series expansion of the function[38]

$$(1 - 2tx + x^2)^{-\lambda} = \sum_{j=0}^{\infty} C_j^\lambda(t) x^j. \tag{A.2}$$

Integrating then over r' and r, one ends up with

$$J_Q^{(2)} = \frac{\pi}{1 + b^2 - 2Qb^2} \sum_{j=0}^{\infty} \frac{1}{2 - 4Qb^2 + j} \int_0^{2\pi} \mathrm{d}\varphi\, C_j^{-2b^2}(\cos\varphi). \tag{A.3}$$

Gegenbauer polynomials with the cosine argument are expressible as[38]

$$C_j^\lambda(\cos\varphi) = \sum_{\substack{k,l=0 \\ (k+l=j)}}^{j} \frac{\Gamma(k+\lambda)\Gamma(l+\lambda)}{k!\,l!\,[\Gamma(\lambda)]^2} \cos(k-l)\varphi, \tag{A.4}$$

which implies

$$\int_0^{2\pi} \mathrm{d}\varphi\, C_{2j+1}^\lambda(\cos\varphi) = 0, \tag{A.5}$$

$$\int_0^{2\pi} \mathrm{d}\varphi\, C_{2j}^\lambda(\cos\varphi) = 2\pi \left[\frac{\Gamma(j+\lambda)}{j!\,\Gamma(\lambda)} \right]^2. \tag{A.6}$$

The consideration of these relations in Eq. (A.3) leads to

$$J_Q^{(2)} = \frac{\pi^2}{1 + b^2 - 2Qb^2} \sum_{j=0}^{\infty} \frac{1}{1 - 2Qb^2 + j} \left[\frac{\Gamma(j - 2b^2)}{j!\,\Gamma(-2b^2)} \right]^2. \tag{A.7}$$

Rewriting in Eq. (A.3) the jth ratio $1/(2-4Qb^2+j)$ as the integral $\int_0^1 \mathrm{d}r\, r^{1-4Qb^2+j}$, the summation over j in Eq. (A.7) can be expressed as

$$\sum_{j=0}^{\infty} \frac{1}{1-2Qb^2+j} \left[\frac{\Gamma(j-2b^2)}{j!\Gamma(-2b^2)} \right]^2 = \frac{1}{\pi} \int_{r<1} \mathrm{d}^2 r\, r^{-4Qb^2} |\mathbf{1}-\mathbf{r}|^{4b^2} \tag{A.8}$$

The integral on the rhs of Eq. (A.8) resembles the Dotsenko-Fateev one $j_1(-Qb^2, b^2, b^2)$ [see the definition (2.20)], however, the domain of integration is restricted to the unit disc. It is trivial to show that

$$j_1(-Qb^2, b^2, b^2) = \int_{r<1} \mathrm{d}^2 r \left[r^{-4Qb^2} + r^{4(Qb^2-b^2-1)} \right] |\mathbf{1}-\mathbf{r}|^{4b^2}. \tag{A.9}$$

In the special case of equal exponents $-4Qb^2 = 4(Qb^2-b^2-1)$, i.e., when $Q = Q_c + 1/2 \equiv Q_1^*$, one finds that

$$j_1(-Q_1^* b^2, b^2, b^2) = 2 \int_{r<1} \mathrm{d}^2 r\, r^{-4Q_1^* b^2} |\mathbf{1}-\mathbf{r}|^{4b^2}. \tag{A.10}$$

The comparison of this relation with Eq. (A.8) implies

$$\sum_{j=0}^{\infty} \frac{1}{1-2Q_1^* b^2+j} \left[\frac{\Gamma(j-2b^2)}{j!\Gamma(-2b^2)} \right]^2 = \frac{1}{2\pi} j_1(-Q_1^* b^2, b^2, b^2). \tag{A.11}$$

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